Second-Order Asymptotics of the Continuous-Time Poisson Channel

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Summary of Results

- The **optimal second-order coding rate** for the continuous-time Poisson channel is derived.
- This is the first instance of the second-order asymptotics for a **continuous-time** channel.

Block Code for Poisson Channel

Figure 1: Coding scheme for Poisson channel

A (T, M, A, σ)-code (ϕ, ψ) consists of

- an encoder $\phi : \{1, 2, \dots, M\} \to \mathcal{W}(T, A, \sigma)$
- a decoder $\psi : \mathcal{S}(T) \to \{1, 2, \dots, M\}$

Moreover, it is called a $(T, M, A, \sigma, \varepsilon)_{\mathrm{avg}}$-code if

$$\frac{1}{M}\sum_{m=1}^{M} \mathbb{P}\{\psi(\nu) = m \mid \lambda = \phi(m)\} \ge 1 - \varepsilon,$$

where λ is the r.v. induced by ϕ with uniform messages.

Denote by $M^* = M^*(T, A, \sigma, \varepsilon)$ the maximum number $M \in \mathbb{N}$ such that a $(T, M, A, \sigma, \varepsilon)_{\mathrm{avg}}$-code exists.

Main Result

As $T \to \infty$, it holds that

$$\log M^* = T\, C^* + \sqrt{T\, V^*}\, \Phi^{-1}(\varepsilon) + O(\log T), \qquad (1)$$

where the **Poisson channel capacity** C^* is given by

$$\begin{cases} C^* \overset{\mathrm{def}}{=} A\left((1-p^*)\, s \log \frac{s}{p^* + s} + p^*(1+s) \log \frac{1+s}{p^*+s} \right), \\ s \overset{\mathrm{def}}{=} \frac{\lambda_0}{A} \qquad \text{(ratio of dark current } \lambda_0 \text{ to PPC } A\text{)}, \\ p^* \overset{\mathrm{def}}{=} \min\{\sigma, p_0\} \qquad \text{(role of CAID)}, \\ p_0 \overset{\mathrm{def}}{=} \frac{(1+s)^{1+s}}{s^s\, e} - s, \end{cases}$$

and the **Poisson channel dispersion** V^* is given by

$$V^* \overset{\mathrm{def}}{=} A\left((1-p^*)\, s \log^2 \frac{s}{p^*+s} + p^*(1+s) \log^2 \frac{1+s}{p^*+s} \right),$$

and $\Phi^{-1}(\cdot)$ stands for the inverse of Gaussian CDF.

- The first-order C^* has already been proven [1, 2, 3].
- The second-order $\sqrt{V^*}\, \Phi^{-1}(\varepsilon)$ is **our contribution**.

Mathematical M … Channel

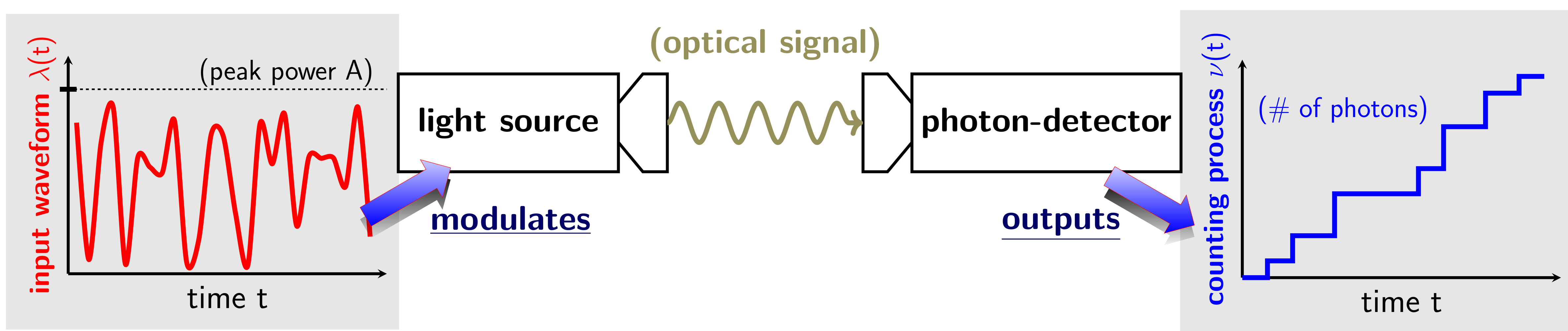


Figure 2: An optical communication with direct photon-detector.

Input Waveform Modulates Light

- integrable function $\lambda(\cdot)$ defined on time block $[0, T)$;
- with **peak power constraint (PPC)** $(A > 0)$:
$$0 \le \lambda(t) \le A \qquad \forall t \in [0, T);$$
- with **average power constraint** $(0 \le \sigma \le 1)$:
$$\frac{1}{T}\int_0^T \lambda(t)\, dt \le \sigma A.$$

Denote by $\mathcal{W}(T, A, \sigma)$ the set of input waveforms $\lambda(\cdot)$.

Output is Poisson Counting Process

The Poisson counting process $\{\nu(t)\}_{0 \le t < T}$ is given by

$$\mathbb{P}\{\nu(0) = 0\} = 0,$$
$$\mathbb{P}\{\nu(t+\tau) - \nu(t)\} = \frac{e^{\Lambda}\Lambda^k}{k!}$$

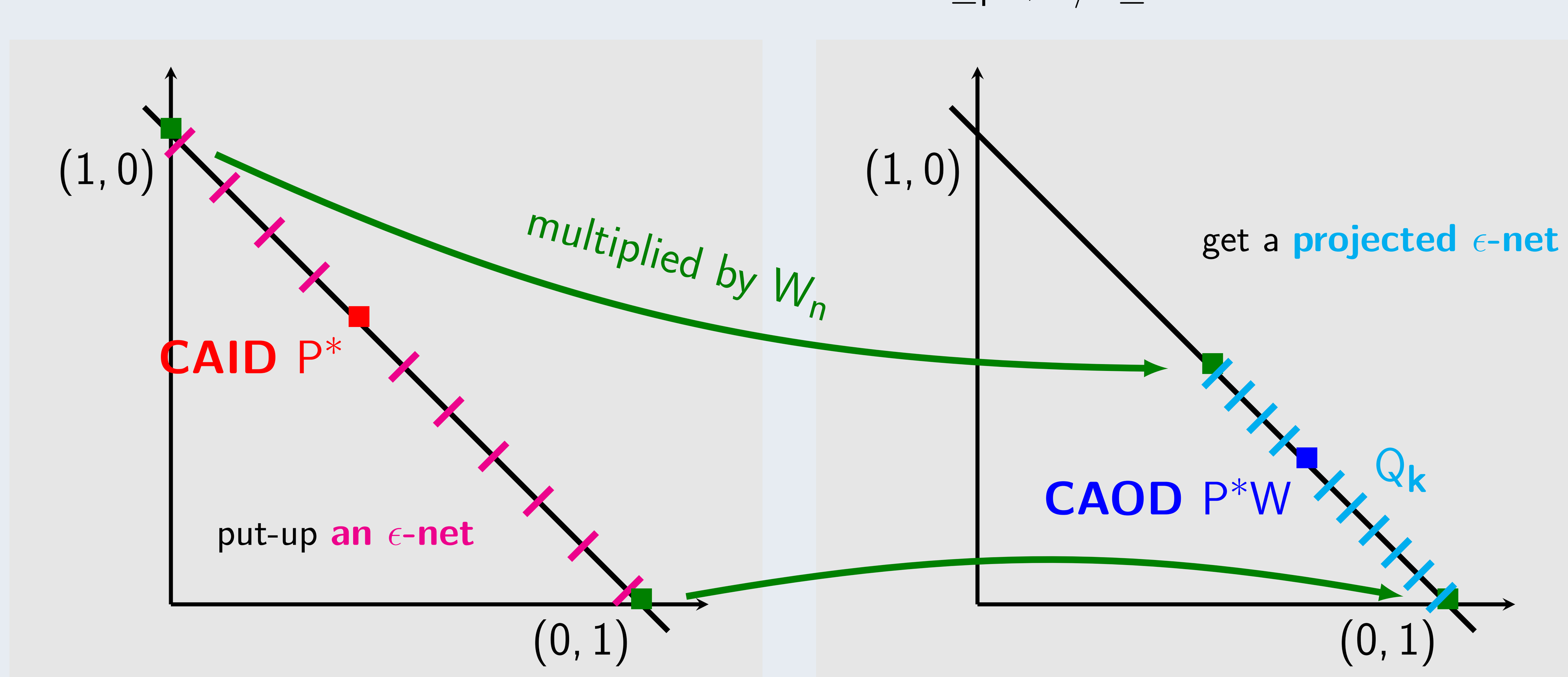
for each $t, \tau \in \mathbb{R}_{\ge 0}$ and $k \in \mathbb{Z}_{\ge 0}$, where Λ is given by

$$\Lambda \overset{\mathrm{def}}{=} \int_t^{t+\tau} \left(\lambda(u) + \lambda_0 \right) du,$$

and $\lambda_0 \ge 0$ is the **dark current** (background noise).
Denote by $\mathcal{S}(T)$ the set of counting processes $\nu(\cdot)$.

Idea of Constructing Output Distribution Q_n on $\{0, 1\}^n$

$$Q^{(n)}(\mathbf{y}) \overset{\mathrm{def}}{=} \frac{1}{3}\prod_{i=1}^{n} P^*_{[-\kappa]} W_n(y_i) + \frac{1}{3}\prod_{i=1}^{n} P^*_{[\kappa]} W_n(y_i) + \frac{1}{3F} \sum_{\substack{m=-\infty: \\ 0 \le p^* + m/T \le 1}}^{\infty} e^{-\gamma m^2/T} \prod_{i=1}^{n} P^*_{[m/T]} W_n(y_i) \qquad (2)$$



We substitute this output distribution $Q^{(n)}$ into the symbol-wise meta converse bound [4]:

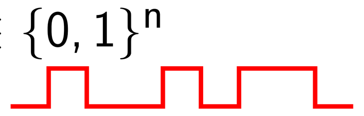
$$\log M^*(W_n^n, \varepsilon + \epsilon_n) \le \max_{\mathbf{x} \in \{0,1\}^n} \sup \left\{ R \in \mathbb{R} \,\middle|\, \mathbb{P}\left\{ \log \frac{W_n^n(Y^n \mid X^n)}{Q^{(n)}(Y^n)} \le R \,\middle|\, X^n = \mathbf{x} \right\} \le \varepsilon + \epsilon_n + \eta \right\} + \log \frac{1}{\eta}.$$

Proof Ideas of Main Result (1)

- In both direct and converse parts, we employ **Wyner's discretization** of the Poisson channel [3].
- In direct part, Feinstein's lemma [5] is used with constant composition code.
- In converse part, a **novel output distribution** $Q^{(n)}$ for the quantized channel is constructed, see (2).

Wyner's Discretization Argument

Figure 3: Wyner's discretization of Poisson channel

- input sequence $\mathbf{x} = (x_1, \dots, x_n) \in \{0, 1\}^n$ (converted to a square wave $\lambda(t)$: )
- output sequence $\mathbf{y} = (y_1, \dots, y_n) \in \{0, 1\}^n$ (obtained by quantizing the counting process $\nu(t)$)

Given a discretization level $n \ge 1$, we get

$$W_n^n(\mathbf{y} \mid \mathbf{x}) \overset{\mathrm{def}}{=} \prod_{i=1}^{n} W_n(y_i \mid x_i),$$

where $W_n : \{0, 1\} \to \{0, 1\}$ is given by

$$W_n(1 \mid x) \overset{\mathrm{def}}{=} \begin{cases} s\, A\, \Delta_n\, e^{-sA\Delta_n} & \text{if } x = 0, \\ (1+s)\, A\, \Delta_n\, e^{-(1+s)A\Delta_n} & \text{if } x = 1 \end{cases}$$

with $\Delta_n = T/n$.

References

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Full Paper is Available at:

https://arxiv.org/abs/1903.10438